Bamboozlers: Encouraging Mathematical Thinking Through Puzzles and Paradox

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Introduction

According to Aristotle, a *metaphor* involves taking something appropriate to one class of things and applying it to something alien. In "Essay on what I think about most", the poet and classicist Anne Carson reflects that metaphor teaches us (Carson, 2000, quoted in Kondratieva, 2007):

> *Not only that things are other than they seem,*
> *and so we mistake them,*
> *but that such mistakenness is valuable.*
> *Hold onto it, Aristotle says,*
> *there is much to be seen and felt here.*
> *Metaphors teach the mind*
> *to enjoy error*
> *and to learn*

This view of metaphor is somewhat akin to paradox, (meaning in Greek "contrary to expectation", or more colourfully, "beyond belief") and relates well to the enjoyment experienced in tackling the obvious mistakenness of a paradox, and the satisfaction of a resolution.

Similarly, students need to be comfortable with error as a natural part of learning. Puzzles and paradoxes provide teachers with a wonderful opportunity to encourage and reward working through error, and exploring the consequences of ideas – both right and wrong. This is an important aspect of real-world mathematical activity. Concerns that a student who finds ordinary classwork mathematics confusing will just view a paradox as confirmation that maths "doesn't make sense" are overstated. With suitable guidance and structure, discussion and collaboration, the challenge and excitement inherent in a paradox encourages the celebration of error and development of techniques to overcome it, thus subtly teaching the nature and rewards of mathematical thinking.

A Fibonacci bamboozlement

I once started a lesson by inviting the class to join me in "doing something impossible", and proceeded to guide them through a puzzle that first appeared over 150 years ago – the dissection of an 8×8 square into the 4 pieces shown in Figure 1, and the subsequent rearrangement to form the 5×13 rectangle shown in Figure 2.

Such a rearrangement is clearly invalid since area is not preserved – somehow an extra square has appeared. Frederickson (2003, chapter 23: *Cheated, Bamboozled, and Hornswaggled*) calls such impossible rearrangements *bamboozlements*.

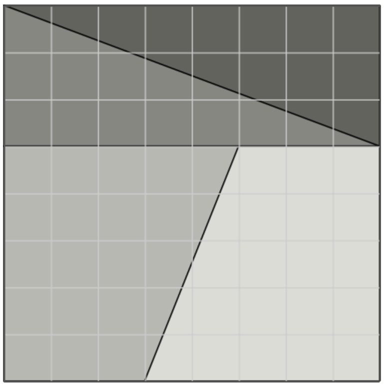

Figure 1: Dissection of a 8x8 square.

The class first worked on this problem as a paper-and-scissors activity, before following up with a web app from the Geometry section of The Mathenæum (http://thewessens.net/maths) to visualise more accurately this case and other variants. This activity is both fascinating and fruitful, and naturally leads to discussion of measurement and area, Pick's theorem for calculating areas on grids (or maps), models of growth, the Fibonacci series in mathematics and in nature, the Golden Ratio, and approximation of irrational numbers.

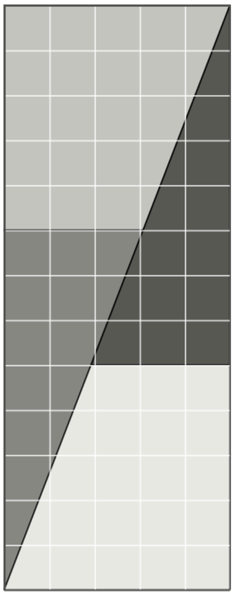

Figure 2: Rearrangement into a 5x13 rectangle.

The attachment the students developed to both the word and concept of a "bamboozler" led to things of this nature becoming a regular feature of lessons, and a great source of wonder and fun.

Remembering what it's like to learn

The ordered presentation of mathematics in a curriculum document or textbook implies learning follows a smooth, steady path. The reality is better understood as comprised of periods of uncertainty and confusion, followed by sudden advances in comprehension (Tall, 1977). A rigid, procedural approach to either teaching or learning mathematics can deliver short-term gain, while hindering the development of deeper, relational understanding. An alternative is to recognise the important role of confusion, and incorporate it into the process in a manner where it generates interest rather than stress.

A perennial obstacle in teaching is remembering how things appeared when first encountered – before they were learned. By going through the following construction, somewhat in the manner of a classroom lesson, we can try and connect with the mindset of a student confronted with a new mathematical concept.

Consider the following number:

3.14159265358979323846264338327950288
41971693993751058209749445923078164
0628620899862803482534211706798...

Of course you recognised π from just the first few digits, but I have included many more in order to emphasise the fact that it goes on forever. It is irrational (in fact transcendental), but we are quite comfortable with that fact and happy to use π in many calculations.

But what about this number?

...99999999999999999

Once again we have an infinitely long string of digits, but this time the infinite extension is to the left! Although we are quite happy with decimal digits extending infinitely to the right of a decimal point, the presentation of this number doesn't sit comfortably at all.

Perhaps it is meaningless? Or maybe it equals infinity (whatever that might mean)?

In an effort to understand this strange number, let's try adding 1 to it, using the normal rules of integer addition.

$$\begin{array}{r} ...99999999999999999 \; + \\ 1 \\ \hline ...00000000000000000 \end{array}$$

The result is an infinite string of zeroes. That also looks odd, but after applying the usual practice of not writing leading 0s, it simply reduces to 0, implying

$$...99999999999999999 = -1.$$

Now that was unexpected.

Is it just a fluke? Let's multiply it by 2, again using the normal rules of integer arithmetic, and see how robust this identification is.

$$\begin{array}{r} ...99999999999999999 \times \\ 2 \\ \hline ...99999999999999998 \end{array}$$

We've generated a new number, still infinitely long to the left, but now ending in 8. It is easy to see that adding 2 to this new number results in 0, and so we can say ...99999999999999998 = −2, and what we've just carried out is simply a strange example of 2 × −1 = −2. Could working with these unusual numbers in this way actually be consistent?

Yes, in fact it is. Just like the normal integers, these numbers form a ring (*i.e.* a group under addition that also supports multiplication and has a multiplicative identity) and are known as the *decadics* (Michon, 2015).

Every integer has a decadic representation (positive integers are unchanged, and negative integers can be constructed by subtracting from …00000 or multiplying by …99999), but they include more than just the integers.

For example:

$$\begin{array}{r} ...66666666666666667 \times \\ 3 \\ \hline ...00000000000000001 \end{array}$$

and so ...66666666666666667 = ⅓, and similarly, ...33333333333333334 = ⅔. It is evident that at least some rational numbers are included.

If we try subtracting ...666666666666667 (*i.e.* ⅓) from 0, we get

$$\begin{array}{r} ...00000000000000000 \; - \\ ...66666666666666667 \\ \hline ...33333333333333333 \end{array}$$

and so …33333333333333333 = −⅓. Compare this with the decadic version of ⅔ above – the only difference is the final digit, yet the value has changed from positive to negative! You might like to try multiplying by ...9999 to confirm the relation.

I hope you are finding this both remarkable and confusing (or if you already were familiar with these numbers that you remember your initial confusion). We have just used ordinary arithmetic, in quite an ordinary way, but far from feeling like we've taken a small step forward, we seem to have constructed something crazy, almost unthinkable!

I suspect these are emotions our students experience all too frequently in their maths classes: e.g. when starting out with negative numbers, fractional indices, infinitesimals etc.

Cognitive conflict

Being confronted with something that seems to "break the rules", even with a secure understanding of the underlying mathematics, produces a sense of internal conflict. We question ourselves – could it really be true? Is it a trick? Do I really know what I thought I knew? This is a state of *cognitive conflict,* and is central to Piaget's theory of learning (Piaget, 1985, reviewed in Cantor, 1983). Cognitive conflict is regarded as a state in which learners become aware of inconsistent or competing ideas, and experience *disequilibration*. Their desire to alleviate this state motivates intensive thinking and critical examination of existing knowledge in an effort to incorporate the new information (*equilibration*).

To be useful in teaching, the conflict experienced by the learners must be appropriate and carefully managed. If the conflict is simply ignored by a student, or otherwise fails to motivate the search for a resolution, the experience will merely reinforce the impression that mathematics is too hard, or that they don't like it. However, appropriate conflict, based on an already secure aspect of knowledge and within the student's ability to resolve, can be quite productive. Two example studies are those of Irwin (1997), who discusses the use of conflict to promote understanding in the context of learning decimals, and Fujii (1987), in the context of learning inequalities.

Counter-examples as generators of cognitive conflict

The problems that result from learning without deep understanding are well documented. Reliance on formulas and memorised rules without proper understanding of their basis and limits of applicability leads to weak mathematical thinking, error, and unnoticed contradiction. Much of the research into cognitive conflict and mathematical learning is directed towards the development of a theoretical framework (e.g. Tall *et al.*, 2014). The focus of this paper is more practical, sharing some experiences of exploiting cognitive conflict in encouraging critical thinking and mathematical understanding.

Counter-examples are the most obvious and easily constructed means of introducing conflict into the teaching and learning environment, and help get students used to explicitly considering the domain of applicability of both existing and new knowledge. It is worth emphasising that simply holding inconsistent ideas does not in itself imply cognitive conflict. The role of the teacher is to generate explicit awareness of the conflict, and then use this awareness to help the learner achieve a resolution and thus remedy misconceptions (Zazkis & Chernoff, 2008).

Klymchuk (2012) employed conflict to teach calculus, and asked his students to construct counter-examples themselves. He noted that they were uncomfortable since the task was not directly algorithmic or procedural, but required mathematical thinking skills similar to real-world situations where some conjecture or hypothesis is under consideration. This is genuine mathematical work – real life without being contrived or condescending. He writes "*Many students are used to concentrating on techniques, manipulations and familiar procedures without paying much attention to the concepts, conditions of the theorems, properties of the functions, nor to the reasoning and justification behind them*" and notes that exposing students only to "nice" functions and "good" examples invites misconceptions by assuming the known properties to be implicit in other contexts. He further reports that the vast majority of students (92%) reported that the conflict-based method was very effective and made learning mathematics more challenging, interesting and creative.

Introducing paradoxes

Paradoxes are counter-examples *par excellence*. They pack more punch, generate more surprise, and maximise the motivational aspect of conflict teaching (Kleiner & Movshovitz-Hadar, 1994; Mamolo & Zazkis, 2008). Paradoxes captivate, amuse, challenge, exasperate and motivate. It is hard for a student to remain passive in the face of such stimulation. They provide the opportunity to participate in debate, with each other and with the teacher, and encourage resilience in the face of obstacles and a tolerance of confusion and error.

While a great deal of mathematics makes intuitive sense, there is also much that is quite removed from the everyday world, even at a basic school level – e.g. objects of fewer than three dimensions, perfect geometrical shapes, irrational numbers *etc*. Failing to recognise this and attempting to connect every mathematical idea directly to something in the "real world" risks reducing mathematical beauty and perfection to little more than measurement and accounting. Lockhart (2009) emphasises the escapism inherent in mathematical curiosity and play (see also Lahme & McDonald, 2006), and researchers Dubinsky & Yiparaki (2000) and Mamolo & Zazkis (2008) have found that everyday, practical intuition often hinders functioning in a new mathematical field.

Mamolo & Zazkis (2008) considered that cornucopia of paradox, *infinity*, a concept that has intrigued minds at least since Zeno of Elea highlighted some of its inherent paradoxes in 450 BC, and recommend strategies that explicitly help students separate their realistic and intuitive considerations from conventional mathematical ones. For example, infinity is not just some "really large number", and its fundamentally different nature is effectively demonstrated by a variety of paradoxes.

Probability is another area that is especially rich in paradox (Leviatan, 2002). Klymchuk (2011) studied using paradox in teaching probability, and found it to be effective in capturing student attention and forcing sustained, deeper engagement. He emphasised the importance of this style of thinking in casual everyday conditional reasoning, and his students were very positive about the usage of paradoxes and counterexamples.

Further enriching the context can be achieved through relating paradoxes and their resolution to the progress of mathematics throughout history. Kleiner & Movshovitz-Hadar (1994) give a wonderful account of paradox clarifying important concepts and driving mathematical advancement through examples such as:

- **Irrational numbers** – The Pythagoreans in the 6th century BC and the inability to represent $\sqrt{2}$ with whole numbers.
- **Negative numbers** and implications for questions of magnitude: e.g. if $\frac{1}{-1} = \frac{-1}{1}$ then the ratio of a greater quantity to a lesser quantity is equal to the ratio of a lesser quantity to a greater quantity.
- **Imaginary numbers** – the formula for solving cubic equations involves intermediate roots of negative values, but still gives correct real answers.
- **Series** and various paradoxes of convergence, e.g. Riemann's proof that you can rearrange the alternating harmonic series
$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots + (-1)^{n-1}\frac{1}{n} + \cdots$$
to give any desired result.
- **Curves and dimension** – e.g. the *Koch Snowflake* fractal has infinite perimeter but finite area.
- **Infinite Sets –** Cantor's proof that infinite sets can have different sizes, and Russell's paradox (i.e. if set $\mathcal{R} = \{x | x \notin x\}$ then $\mathcal{R} \notin \mathcal{R} \Leftrightarrow \mathcal{R} \in \mathcal{R}$).

Presenting paradoxes in their historical context illustrates to students that mathematics is not just the serial acquisition of a predetermined set of rules, but that faulty logic and erroneous arguments are a frequent occurrence in a dynamic, lively, problem driven, exploratory field of study. Maths is as much about questions as about answers. Learning maths is about developing the ability to generalise, carefully consider reasoning and boundaries, and construct examples or counterexamples.

Bamboozling with algebra

An incomplete understanding of the bounds of applicability of algebraic manipulations is a common source of error. Students can latch on to an algebraic process as a purely mechanical recipe, and apply it without awareness of the underlying mathematics. The examples in this section are not true paradoxes, but paradoxical results due to subtle but common errors of algebra (Barbeau 2000 & 2013 have many examples). As such they are an amusing and valuable way to demonstrate the consequences of improper algebraic manipulations, and so remind students of the need to always think about the mathematical basis of what they are doing.

"Proving" 1 = 2

I used this false proof when teaching the difference of two squares binomial identity $a^2 - b^2 = (a+b)(a-b)$.

Let $a = 1$ and $b = 1$, then clearly

$$\begin{aligned} b &= a \\ ab &= a^2 \\ ab - b^2 &= a^2 - b^2 \\ b(a-b) &= (a+b)(a-b) \\ b &= a+b \\ 1 &= 2 \end{aligned}$$

The students were blown away by the result, and of course most distrusted the step where $a^2 - b^2 = (a+b)(a-b)$ was used, since that was new. But this is ideal – it forced them to study the identity and convince themselves of its validity, and then look elsewhere for the problem. Eventually the divide-by-zero flaw was uncovered, itself an important aspect of algebraic manipulation and awareness.

"Proving" 2 = 4

This "proof" is similar to the one above, and gives students the opportunity to engage with the perfect square binomial identity while reinforcing the rules for solving equations involving squares.

We start with an obvious equality and carry out some basic manipulations (always doing the same thing to each side) before reaching a state where we can apply the identity $(a-b)^2 = a^2 - 2ab + b^2$.

$$-8 = -8$$
$$4 - 12 = 16 - 24$$
$$2^2 - 2 \times 2 \times 3 = 4^2 - 2 \times 4 \times 3$$
$$2^2 - 2 \times 2 \times 3 + 3^2 = 4^2 - 2 \times 4 \times 3 + 3^2$$
$$(2-3)^2 = (4-3)^2$$
$$2 - 3 = 4 - 3$$
$$2 = 4$$

Not as neat as the 1 = 2 example, but still quite nice, and with its own important lessons.

An equation you can't get wrong.

Solving equations is frequently taught in a very mechanical way – identify the *form* of the equation, then apply the appropriate designated process. Explaining the apparent paradox in the solution of this equation encourages students to look more deeply and consider a graphical interpretation as well, before simply proceeding to solve.

Suppose some students need to solve the following equation for x (from an example in Barbeau, 2013)

$$\frac{4x-12}{3} = \frac{2x-6}{5}.$$

The first student multiplies through by 5 and then by 3 to get

$$20x - 60 = 6x - 18$$

which is easily converted to

$$14x = 42$$

and so finds $x = 3$.

A second student also multiplies through by 5, but just forgets about the 3 to get

$$20x - 60 = 2x - 6$$

which becomes

$$18x = 54$$

and also finds the solution $x = 3$.

Another student only multiplies through by 3 and forgets about the 5, while yet another just ignores the denominators entirely. These two also find $x = 3$.

We seem to have an equation that resists all attempts to get it wrong!

Explaining this requires careful thinking about what the equation represents, what the standard solution finding manipulations mean, and situations where they are not appropriate.

Bamboozling with large numbers

Large numbers are genuine objects of fascination, and thus a valuable tool for increasing engagement in a classroom. They arise in many mathematical and scientific contexts, and in particular through exponential growth. Yet even for those of us quite used to working with exponential functions, the sheer magnitude of their growth can be mind-boggling.

Puzzles with 4 numbers

A fun puzzle is to choose a number between 1 and 10, and then use that number exactly four times in an expression involving basic mathematical operations, e.g. at least $+$, $-$, $\times$, $\div$ but also possibly $\sqrt{\ }$ and n^n, and see what values can result.

With four 2s, all values from 1 to 10 except 7 are easily constructed. I have frequently given this task to a class – getting a result of 9 is a good challenge. Then, to lead into a discussion of exponential growth, we think about what is the largest possible value.

Encouraging creative thinking is important in this task, and usually before long someone will incorporate juxtaposition of the 2s, allowing the use of 22, 222, and 2222. This is where things get interesting, since they need to grapple with questions like is $2^{222} > 22^{22}$ and why.

Learning that the exponent is more significant than the base in producing the largest possible value is the important outcome, and so the solution is $2^{2^{22}} = 2^{4194304}$. Even better is the fact that, although this number is too big for their calculators, some basic mathematical thinking allows us to comfortably work with it. Its size can be estimated by noting that $2^3 = 8 \approx 10$, so every three 2s approximately results in an extra digit in the answer. This leads to a quite incredible estimate of $4194304 \div 3 \approx 1.4$ million digits! Close enough to the actual value of 1262612 digits, but maximum effect is achieved by showing all digits (http://thewessens.net/ClassroomApps/Main/misc/bignum.txt).

Reaching this point has already involved quite a lot of interesting and important mathematical thinking and activity, but where appropriate an obvious extension is trying the same thing with 3s. The largest value is $3^{3^{33}} = 3^{5559060566555523}$, and this time, noting that $3^2 = 9 \approx 10$, we see that every two 3s approximately results in an extra digit in the answer, and so our number is approximately $2\frac{1}{2}$ thousand million million digits long. It is worth emphasising – this is not the size of the number, but the *number of digits* in the number. One million has only 6 digits, a billion only 9, a trillion only 12. This number has quadrillions of digits! It is estimated that the total number of atoms in the universe is roughly 10^{80} — a measly 80 digit number. A number with over a million digits, such as the earlier seen $2^{2^{22}}$ is incredibly large, but can still be easily printed out. If we were to print $3^{3^{33}}$ one hundred thousand digits to a page and share the pages among the Earth's population, every man, woman and child on Earth would need to hold 4 pages.

And if just moving from 2s to 3s makes this much of a difference, what happens when you get all the way up to 9s?

Now, many students will be familiar with a *googol,* i.e. 1 followed by 100 zeroes. Some may also know that a *googolplex* is 1 followed by a googol zeroes. (The story of where these names came from is a good one and worth mentioning.) A googolplex is unimaginably huge. Even if the entire universe was filled with microscopically small digits, the number written would be nowhere near a googolplex. And even if we produced billions of digits a second, for billions of times the age of the universe, we would barely get started printing it out. We would need to write maybe 10 digits every second on every atom in the universe for the entire lifetime of the universe to get close to writing out a googolplex. But, using just four 4s, we get a number, $4^{4^{4^4}} \approx 10^{10^{154}}$, that absolutely dwarfs a googolplex. Writing out this number is like writing a googolplex one million trillion trillion trillion trillion times! To go up to using four 9s sends these values beyond any meaningful bounds. Indeed, even the number that describes the number of digits in a power tower of four 9s is itself 370 million digits long. Think on that for a moment – a 370 million digit number just to describe the number of digits!

Large numbers have wonderful powers of amazement, and it is exciting for students to discover how they can be generated so easily, and to think of ways to try and appreciate their magnitudes.

Bamboozling with infinity

Moving on from very large numbers, we naturally come to infinity. But infinity is not just a very big number. Although it is common to think about infinity in this way, doing so without proper care leads to many paradoxes. Nevertheless these paradoxes can be quite illuminating, and certainly enjoyable.

Grandi's series

At the start of the 18th century, the Italian mathematician Guido Grandi studied the infinite series

$$S = 1 - 1 + 1 - 1 + 1 - 1 + \cdots.$$

An obvious way to determine the value of S is to group the 1s in pairs and get

$$S = (1-1) + (1-1) + \cdots = 0 + 0 + \cdots = 0.$$

That seems perfectly reasonable, even trivial, but it is just as reasonable to keep the first 1 separate and group the others to get

$$\begin{aligned} S &= 1 + (-1+1) + (-1+1) + \cdots \\ &= 1 + 0 + 0 + \cdots \\ &= 1\,. \end{aligned}$$

This is a very surprising result – the same sum seems to equal both 0 and 1.

But it gets even stranger. We can move a few extra 1s to the front, and get

$$\begin{aligned} S &= 1 + 1 + 1 + 1 + (-1+1) + (-1+1) + \cdots \\ &= 4 + 0 + 0 + \cdots \\ &= 4\,. \end{aligned}$$

This works because even though we've brought some 1s to the front, we never run out of positive and negative terms to pair up afterwards, so all the other terms still cancel. It seems that this series can be made to equal any integer simply by rearranging the terms.

Trying a more algebraic approach, we can write

$$S = 1 - 1 + 1 - 1 + 1 - 1 + \cdots.$$

Adding a single 0 at the front can't change the value of the sum, so we also have

$$S = 0 + 1 - 1 + 1 - 1 + 1 + \cdots.$$

Then adding term by term gives

$$\begin{array}{r} S = 1 - 1 + 1 - 1 + \cdots \ + \\ S = 0 + 1 - 1 + 1 - \cdots \\ \hline 2S = 1 + 0 + 0 + 0 + \cdots \\ \hline \end{array}$$

and so $2S = 1$, and therefore $S = \frac{1}{2}$.

That's the strangest result yet. We have just shown, somehow, that $1 - 1 + 1 - 1 + \cdots = \frac{1}{2}$.

Attempts to deal with this occupied many of the great mathematical minds of the 18th and 19th centuries – Euler, Leibnitz, Abel, various Bernouillis, and if nothing else, Grandi's series makes a great opportunity to discuss the history of maths. Eventually this work led to the formal realisation that infinite series are more than just very long sums and require different rules.

From a modern perspective, this result can be dismissed as simple trickery and used to frame a discussion of divergence, convergence and absolute convergence of infinite series. But that's not the whole picture – there is more mathematics here than meets the eye. In particular, there are different ways to understand *summation* when there is an infinity of terms (implicit in the $\ldots 99999 = -1$ result we saw earlier).

Moving from the finite to the infinite is a fundamental change, and it is not unusual in maths to push things beyond a boundary and then extend their meaning. For example, modular arithmetic and negative indices are two examples where simple, intuitive concepts are extended, in not immediately intuitive ways, to cover a new domain.

There is much of mathematics for students to experience and learn by exploring the boundaries. Indeed, when something that shouldn't work produces something interesting, it is not a sign that it should be avoided, but rather that there is something more to be learned. A bit of experimentation can result in a wonderful journey of discovery.

An infinite number of ping pong balls, and an hour to spare

Infinite sets are prodigious generators of paradoxes, and many are accessible at the level of school mathematics. From very early on students must grapple with the fact that there is a one-to-one correspondence between the set of all positive integers, and the set of all multiples of 10 for example. The following paradox presents a "practical" demonstration of this fact (Morgan 2000).

(The process I will describe involves a *supertask* – that is a task that requires completing an infinite number of steps in a finite amount of time.)

Clark has a superpower, and so is able to complete supertasks. He has before him an infinite supply of ping pong balls and a bag. At 10:00 he places 10 balls in the bag, and then takes 1 out. Then at 10:30 he places a further 10 balls in the bag, and again takes 1 out. He continues this process, but each time waits only half the previous wait time, so the next step is at 10:45, then 10:52:30 etc. Since

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=1$$

the process ends, after an infinite number of steps, at 11:00am exactly. (This can reasonably be demonstrated graphically to students who are unfamiliar with geometric series.)

The question is, how many balls will be in Clark's bag at 11:00am?

Suppose the balls are numbered 1, 2, 3, …. If at each step the next 10 balls are placed, and the 10th one removed, the balls that end up in the bag are precisely those numbered with values that are not multiples of 10. Mathematically, on step n, ball $10n$ is removed. Clearly, according to this argument, there are an infinite number of balls in the bag at 11:00am.

Now suppose that Clark has a friend, Bruce, who is simultaneously carrying out this exercise, but using a slightly different strategy. Bruce places in the bag the next 10 balls (by number) each time, in the same way as Clark, but removes the ball with the smallest number (instead of the last ball added). So ball 1 is removed on step 1, ball 2 on step 2, and ball n on step n. At every step the number of balls in Clark's bag and Bruce's bag is the same, but, amazingly, at 11:00am Bruce's bag is empty! Every ball has been removed – ball 1 on step 1, ball 10 on step 10, ball 567 on step 567 etc. Because there were an infinite number of steps, every number ball was removed.

This is an incredible result. At every instant of time prior to 11:00am, the two bags have the same number of balls. Then, instantly at 11:00am, one bag is empty and the other infinitely full!

This paradox illustrates why subtracting infinity from infinity is undefined – the result depends on how you do it. Specifically, it demonstrates the difference between comparing two infinite sets like this (Bruce's method – leaves none behind):

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 2 & 4 & 6 & 8 & 10 & \ldots \end{array}$$

or like this (Clark's method – leaves an infinite number behind):

$$1 \quad \not{2} \quad 3 \quad \not{4} \quad 5 \quad \not{6} \quad 7 \quad \not{8} \ldots$$

Some extensions:

- It is possible to use a probabilistic argument when the balls are unlabelled to show that the chance of any particular ball remaining in the bag after infinite withdrawals is 0. Just calculate the likelihood of any particular ball surviving 1 step, then N steps, and take the limit $N \to \infty$.
- Imagine there is a gremlin in Clark's bag that sneakily erases the '0' on ball 10 as he removes it, and draws it on ball 1 in the bag making it ball 10. The balls remaining in the bag are now numbered 2 to 10, and Clark has a ball numbered 1 in his hand. The gremlin does the same when Clark removes ball 20 – making it 2 and changing the 2 ball to 20. Now the balls in the bag have the numbers 3 to 20. If this continues, will there still be an infinite number of balls in Clark's bag at 11:00? He is, after all, removing the same balls as before. But if there are balls remaining in the bag, what will their numbers be?
- How can a bag *ever* end up empty if every step 10 balls are added and only 1 taken out?
- Clearly this supertask is physically impossible. Do these paradoxes imply it is also logically impossible?

Bamboozling with probability

For such a simple concept, essentially counting possibilities, probability can be incredibly confusing and is replete with counterintuitive results and paradoxes. The examples I present below are chosen because they relate directly to everyday situations and misconceptions.

Competing with unusual dice

Burger & Starbird (2010) describe a game involving a set of 4 unusual dice – one die has two 6s and four 2s, another has three 5s and three 1s, a third has four 4s and two blank faces, and the last die has a 3 on all faces. The game starts with one player choosing one of the four dice to use, followed by the other player choosing one of the remaining three. The players roll against each other, the winner each roll being the player who rolls the largest value, and the winner overall is the player who won the most rolls.

The paradox arises when trying to determine which die is the optimal choice for the first player. If he chooses the 6+2 die, he will be defeated by the all 3s die two-thirds of the time. But if he chooses the all 3s die, he will be defeated by the 4+blank die two-thirds of the time. Similarly, if he chooses the 4+blank die, he will be defeated by the 5+1 die two-thirds of the time. Finally, if he chooses the 5+1 die, he will be defeated by the 6+2 die two-thirds of the time, and we are back where we started.

A similar situation may arise in preferential voting. Suppose in an election there are 3 candidates, call them A, B and C, and 300 voters. After counting the ballots it is found that:

- 100 voters ranked A > B > C
- 100 voters ranked B > C > A
- 100 voters ranked C > A > B

Which candidate won?

Well, each of A, B and C received 100 first preference votes, so that doesn't decide anything.

Since two-thirds of voters preferred A over B, and two-thirds of voters preferred C over A, it seems C should be declared the winner. But two-thirds of voters preferred B over C! This is called a *Condorcet cycle* (after a 1785 study on majority decisions by the French philosopher and mathematician Nicolas de Condorcet). It should also feel quite familiar to anyone who has played a game of rock/paper/scissors.

Paradoxes of conditional probability

Conditional probability can be quite counterintuitive, a fact that leads to many paradoxical results, made all the more interesting since they are readily connected to simple situations and everyday events. The fallibility of human judgement when confronted with randomness and dependent events was the subject of a popular account by Taleb (2005).

Bayes' theorem

$$P(A|B) = \frac{P(A)}{P(B)} P(B|A)$$

describes the connection between the independent probabilities for events A and B, and the conditional probabilities of A occurring given B has occurred, and the reverse. Intuition often leads to error when there are significant differences in magnitude between the terms.

The base rate fallacy

The *base rate fallacy* is the tendency to ignore the prevalence of the base case or general condition, and arises most readily in situations where there is testing for a rare event. For example, suppose a "stolen car detector" has 100% accuracy when scanning a stolen car, and 99% accuracy when scanning a non-stolen car – i.e. if it scans a stolen car it always correctly flags it as stolen, and if it scans a non-stolen car, it correctly recognises that it is not stolen 99% of the time. Given that approximately 100 000 cars cross the Sydney Harbour Bridge on any day, of which maybe 10 are stolen, how successful would this 99% accurate device be if deployed to scan them all? Well, all 10 stolen cars will be identified, so that's good. But of the 99 990 not-stolen cars that are scanned, 1% of them, or effectively 1000 cars, will be incorrectly identified as stolen. This is the *base rate*, and shows us that the chance of a flagged car being stolen is actually just $^{10}/_{1010}$ or less than 1%!

The prosecutor's fallacy

A similar problem is the *prosecutor's fallacy*, and occurs because of our tendency to incorrectly interpret the implications when an unlikely event occurs (*DNA Fingers Murderer*, in Paulos, 2013, p.72-3). For example, suppose, in a city of 1 million people, a serious crime has been committed, and DNA evidence indicates that the perpetrator has a particularly rare characteristic, present in only 1 of every 10 000 individuals. Surely that will make it easier to catch the culprit!

Having brought a suspect to trial, the prosecutor declaims to the jury:

The genetic signature of the criminal, found at the scene of the crime, is extremely rare. The chance of someone carrying this DNA is just 1 in 10 000 — only 0.01%. Yet the accused has this DNA, and this alone shows how likely is his guilt!

Fortunately for the accused, there is a mathematician in the jury. She quickly calculates that given there are 1 million people in the city, a frequency of 1 in 10 000 implies the number of people with the particular DNA signature is 100, and 99 of those people are innocent. This is the crucial point: the chance of someone with that DNA being guilty is only 1%. The prosecutor is making an elementary mistake in probability, and must rely on additional evidence if he is to obtain a conviction.

Feedback

The feedback from students regarding the *bamboozlings* was overwhelmingly positive. Some particular comments are shown below:

- *Inspirational*
- *Learnt so much despite the bamboozling*
- *It was really fun*
- *Made me realise maths is so much more than numbers on a page*
- *Thanks for the "out of textbook" lessons*
- *Opened up a new side of maths*
- *These lessons will stay with me for the rest of my life*
- *I began to appreciate your love of maths*
- *You have changed the way I think and approach mathematical questions*

Such responses reinforce the view that including paradoxes and puzzles is a satisfying and rewarding activity for teacher and student alike.

Conclusion

There are many paradoxes, counter-intuitive results and fallacies in mathematics – more than enough to support an enriching, engaging and alternative lesson on any topic. Further, noticing and responding to error guides an ongoing collection of such items. I have presented here only a few examples, but there are many more I would have loved to have included, such as:

- The Koch Snowflake – infinite perimeter, finite area.
- Painting Gabriel's Horn (cover an infinite surface area with a finite volume of paint),
- The Potato Paradox (a counter-intuitive result arising from simple ratios),
- The Two-child Paradox,
- The Birthday Paradox,
- The Monty Hall Paradox,
- Simpson's Paradox,
- Hilbert's Grand Hotel,
- The Barber Paradox,
- Bertrand's Paradox.

Additionally, continued fractions, the logistic map, and other recursive procedures provide rich extension opportunities where basic ideas are extended to generate cognitive conflict and mathematically fruitful subsequent resolution.

These problems entertain, illuminate, and force students to consider the limits of applicability of important concepts, rules and methods. Certainly there are challenging aspects to both teaching and learning this way, but the rewards are well worth the effort.

In summary, paradoxes teach that confusion is both normal and good, they inform us historically, directly address important sources of error, and, most important of all, present mathematics and mathematical thinking in a positive and engaging light.

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Appendix A – *p*-adics

The decadics are not particularly useful numbers since they allow undesirable results, such as non-zero values that multiply to zero.

But the decadics are just one example of these infinitely-leftish numbers. The most useful ones are called *p-adics* (Gouvêa, 1997; Michon, 2015), which are in base p for p a prime.

Formally, a p-adic expansion is defined by the series

$$\sum_{i=n}^{\infty} a_i\, p^i$$

where the sum index i starts from a finite, but possibly negative, n.

The p-adics, written $\mathbb{Q}_p$, form fields, and are completions of the rationals (and as such are an alternative to the usual field of real numbers). That they go beyond the rationals is apparent from the fact you can write in $\mathbb{Q}_7$, for example,

$$\sqrt{7} = \ldots 6421216213,$$

or, even more surprising, in $\mathbb{Q}_5$ the equation $x^2 = -1$ has solution $x = \ldots 0223032431212$. (Square the last n digits in base 5 to see the corresponding line of 4s in the result; the same reasoning as we used earlier shows that in $\mathbb{Q}_5$, $\ldots 44444 = -1$.)

Appendix B – Mathenaeum Activities

Various activities at *The Mathenaeum* were developed to support investigating puzzles and paradoxes. Some of the most relevant ones are listed with links below:

- *Four Twos* http://thewessens.net/ClassroomApps/Main/fourtwos.html
- *Fibonacci Bamboozling* http://thewessens.net/ClassroomApps/Main/dissection.html
- *Circles on Circles:* http://thewessens.net/ClassroomApps/Main/coinroll.html
- *Aristotle's Wheels* http://thewessens.net/ClassroomApps/Main/aristotle.html
- *Quadratic Chaos* http://thewessens.net/ClassroomApps/Main/logistic.html
- *Ants on a Stick* http://thewessens.net/ClassroomApps/Main/ants.html
- *Crazy Dice* http://thewessens.net/ClassroomApps/Main/crazydice.html
- *Dodgeball* http://thewessens.net/ClassroomApps/Main/dodgeball.html